Entanglement Sudden Death as an Indicator of Fidelity in a Four-Qubit Cluster State

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I explore the entanglement evolution of a four qubit cluster state in a dephasing environment concentrating on the phenomenon of entanglement sudden death (ESD). Specifically, I ask whether the onset of ESD has an effect on the utilization of this cluster state as a means of implementing a single qubit rotation in the measurement based cluster state model of quantum computation. To do this I compare the evolution of the entanglement to the fidelity, a measure of how accurately the desired state (after the measurement based operations) is achieved. I find that ESD does not cause a change of behavior or discontinuity in the fidelity but may indicate when the fidelity of certain states goes to .5.

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I. INTRODUCTION

Entanglement is a uniquely quantum mechanical phenomenon in which quantum systems exhibit correlations above and beyond what is possible for classical systems. Entangled systems are thus an important resource for many quantum information processing protocols including quantum computation, quantum metrology, and quantum communication [1]. Much work has been done with respect to the identification and quantification of entanglement as well as explorations of entanglement evolution under a range of possible dynamics [2].

An important area of research is to understand the possible degredation of entanglement under decoherence. Decoherence, unwanted interactions between the system and environment, is the major challenge confronting experimental implementations of quantum computation, metrology, and communication. Decoherence may be especially detrimental to highly non-classical, and hence the most potentially useful, entangled states [3]. A manifestation of the detrimental affects of decoherence on entangled states is entanglement sudden death (ESD) in which entanglement is completely lost in a finite time [4, 5] despite the fact that the loss of system coherence is asymptotic. This aspect of entanglement has been well explored in the case of bi-partite systems and there are a number of studies looking at ESD in multi-partite systems [6, 7, 8, 9, 10, 11]. In addition, there have been several initial experimental ESD studies [12].

The ESD phenomenon is interesting on a fundamental level and important for the general study of entanglement. However, it is not yet clear what the affect of ESD is on quantum information protocols. Are different quantum protocols helped, hurt, or indifferent to ESD? Previous studies along these lines have been in the area of quantum error correction (QEC). An explicit study of the three-qubit phase flip code concludes that this specific code is indifferent to ESD [11]. In this paper I take a first step in studying the affect of ESD on cluster state quantum computational gates. Specifically, I study a four qubit cluster state to see how ESD affects its utility as a means of implementing a general single qubit rotation for measurement based (cluster state) quantum computation. My approach will be to use an entanglement witness, the negativity and bi-partite concurrence as entanglement metrics and compare the behavior of these metrics under the influence of decoherence to the fidelity of the final state after the attempted single qubit rotation. In addition, I will study the entanglement that remains in the cluster state after two measurements and compare it to the fidelity of the state of the two unmeasured qubits.

The cluster state [13] is a specific type of entangled state that can be used as an initial resource for a measurement based approach to quantum computation [14]. A cluster state can be created by first rotating all qubits into the state $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$. Desired pairs of qubits are entangled by applying control phase (CZ) gates between them. In a graphical picture of a cluster state, qubits are represented by circles and pairs of qubits that have been entangled via a CZ gate are connected by a line. A cluster state with qubits arranged in a two-dimensional lattice, such that each qubit has been entangled with four nearest neighbors, suffices for universal QC.

After constructing the cluster state, any quantum computational algorithm can be implemented using only single-qubit measurements along axes in the x-y plane. These processing measurements are performed by column, from left to right, until only the last column is left unmeasured. The last column contains the output state of the quantum algorithm which can be extracted by a final readout measurement. One can view each row of the cluster-state lattice as the evolution of a single logical qubit in time. Two (logical) qubit gates are performed via a connection between two rows of the cluster state. CZ gates in particular are 'built-in' to the cluster state and simple measurement automatically implements the gate. Single qubit rotations can be performed when there is no conncetion between the measured qubit(s) and qubits in another row. In such a case the logical gate implemented by measurement along an angle ϕ in the x-y plane is $X(\pi m)HZ(\phi)$, where H is the Hadamard gate and $Z(\alpha)$ $(X(\alpha))$ is a z- (x-) rotation by an angle α [15]. The dependence of the logical operation on the

outcome of the measurement is manifest in $m = 0, 1$ for measurement outcome $-1, +1$. An arbitrary single qubit rotation can be implemented via three logical single-qubit rotations of the above sort yielding

$$HZ(\alpha + \pi m_\alpha)X(\beta + \pi m_\beta)Z(\gamma + \pi m_\gamma),$$

where (α, β, γ) are the Euler angles of the rotation. For example, by drawing the Euler angles according to the Haar measure, a random single-qubit rotation can be implemented.

As with all quantum computing paradigms, cluster state quantum computation, both during the constuction of the cluster state and during subsequent measurement, are subject to decoherence. We study a four qubit cluster chain, with no interaction between the qubits (beyond the initial conditional phase gates used to construct the cluster state) placed in a dephasing environment fully described by the Kraus operators

$$K_1 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{pmatrix}; \quad K_2 = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{p} \end{pmatrix} \tag{1}$$

where we have defined the dephasing parameter p. When all four qubits undergo dephasing we have 16 Kraus operators each of the form $A_l = (K_i \otimes K_j \otimes K_k \otimes K_\ell)$ where $l = 1, 2, ..., 16$ and $i, j, k, \ell = 1, 2$. Though all of the below calculations are done with respect to p, I implicitly assume that p increases with time, τ, at a rate κ, such that $p = 1 - e^{-\kappa\tau}$ and $p \to 1$ only at infinite times. For now I also assume equal dephasing for all four qubits.

In optical cluster state construction small (few qubit) cluster states are fused together to form larger cluster states [16]. The smaller states must be stored until they are needed and may be subject to decohence (especially dephasing). In other cluster state implementations, where complete two-dimensional cluster states can be constructed in just a few steps [17], any four qubit chain may be attached to at least one other qubit. In this case our results may not be exact.

While entanglement is invariant to single qubit operations, decoherence is not and local operations may play a significant role in the entanglement dynamics of the state. Thus, if a cluster state must be stored in a decohering environment one would ideally like to choose a cluster state representation (within single qubit operations) that has the greatest immunity to the decoherence so as retain as much entanglement as possible. With this is mind a secondary aim of this paper is to study two representations of the four qubit chain cluster state and compare the affects of dephasing on these representations. The first representation of the four qubit cluster state is

$$|C_4\rangle = \frac{1}{2}\left(|0000\rangle + |0011\rangle + |1100\rangle - |1111\rangle\right). \tag{2}$$

This representation minimizes the number of computational basis states having non-zero contribution. The second representation is:

$$|C_{4H}\rangle = H_1 H_4 |C_4\rangle, \tag{3}$$

where H_j is the single qubit Hadamard gate on qubit j. This is the state one would get by initially rotating each qubit into the state $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and applying contolled phase gates CZ_{12}, CZ_{23}, and CZ_{34}. We note that 'connections' between qubits may be added or removed by single qubit rotations (though the entanglement stays constant) thus changing the operation performed via measurement [18].

The four qubit cluster has pure four qubit entanglement. Thus, for example, there is no bi-partite concurrence between any of the two qubits. As an entanglement metric we use the negativity, N, for which we will simply use the most negative eigenvalue of the parital transpose of the density matrix [19]. There are a number of inequivalent forms of the negativity for the four qubit cluster state: the partial transpose may be taken with respect to any single qubit, N_1, or the partial transpose may be taken with respect to two qubits: qubits 1 and 2, N_{12}, qubits 1 and 3, N_{13}, or qubits 1 and 4, N_{14}.

A further method of monitoring entanglement evolution is via the expectation value of the state with respect to an appropriate entanglement witness [20]. Entanglement witnesses are observables with positive or zero expectation value for all states not in a specified class and a negative expectation value for at least one state of the specified class. Entanglement witnesses may allow for an efficient means of determining whether entanglement is present in a state (as opposed to inefficient state tomography). This is especially important for experimental implementations as it may be the only practical means of deciding whether or not sufficient entanglement is present in the system. The entanglement witnesses I use are designed to detect cluster states and will be either $\mathcal{W}_{C4} = \mathbb{1}/2 - |C_4\rangle\langle C_4|$ or $\mathcal{W}_{C4H} = \mathbb{1}/2 - |C_{4H}\rangle\langle C_{4H}|$ depending on the representation [21].

II. ESD IN A FOUR QUBIT CLUSTER STATE

Our first step is to determine at what dephasing strength, p, (if any) the four qubit cluster state exhibits ESD. The final state of the four qubit system after dephasing is given by $\rho_r(p) = \sum_l^{16} A_l |C_r\rangle\langle C_r| A_l^\dagger$ where $r = 4, 4H$. Figure 1 shows the evolution of our chosen entanglement metrics for initial cluster states as a function of p. For the intial state $|C_4\rangle$ the expectation value of the final state after dephaing with respect to the entanglement witness, $\mathcal{W}_{C4}$, is given by $-\frac{1}{4}\left(p^2 - 4p + 2\right)$. Thus, cluster state entanglement can be detected by the entanglement witness for $p < 2 - \sqrt{2} \simeq .586$. Interestingly, the expectation value with respect to the entanglement witness is equal to N_{12}, the most negative eigenvalue of the partial transpose of the final state with respect to qubits 1 and 2, which thus exhibits ESD at the same value. N_1, N_{13}, and N_{14} do not undergo ESD. N_1, the lowest eigenvalue of the partial transpose of the final state with respect to one qubit, is given by $-\frac{1}{4}\left(p^2 - 3p + 2\right)$. The most negative eigenvalues of the partial transpose of the

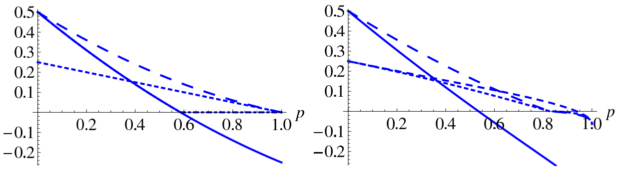

FIG. 1: (Color online) Entanglement evolution as measured by $-\mathrm{Tr}[\mathcal{W}_{\mathrm{r}}\rho_{\mathrm{r}}(\mathrm{p})]$ (solid line), N_1 (large dashed line), N_{12} (chained line), N_{13} (medium dashed line), and N_{14} (small dashed line) for intial states $|C_4\rangle$ (left) and $|C_{4H}\rangle$ (right) as a function of dephasing strength p on all four qubits. For intial state $|C_4\rangle$ there is no ESD for N_1 or $N_{13} = N_{14}$, but ESD is exhibited for N_{12} at $p \simeq .586$. The expectation value of the dephased state with respect to the entanglement witness $\mathcal{W}_{C4}$ is equivalent to N_{12}. For initial state $|C_{4H}\rangle$, $N_1 = N_{12}$ and ESD occurs at $p \simeq .828$. This is the same value for which N_{14} exhibits ESD. ESD for N_{13} is exhibited at $p = .938$. The entanglement witness, $\mathcal{W}_{C4H}$ fails to detect entanglement for $p \gtrsim .535$.

state with respect to qubits 1 and 3 (N_{13}) and 1 and 4 (N_{14}) are four times degenerate and given by $\frac{1}{4}(p-1)$. Non-zero negativity for only some qubit partitions implies the presence of bound entanglement. For the initial state $|C_4\rangle$ under dephasing bound entanglement is present in the state for $p \gtrsim .586$.

For the intial state $|C_{4H}\rangle$ $N_1 = N_{12}$ with the most negative eignevalue of the partial transpose of the final state given by $\frac{1}{16}\left(-4 - 4\tilde{p}^3 + 6p - p^2\right)$, where $\tilde{p} = \sqrt{1-p}$. Both exhibit ESD at $p = -2 + 2\sqrt{2} \simeq .828$. For N_{13} the most negative eigenvalue is given by $\frac{1}{16}\left(-4\tilde{p} + 2p - p^2\right)$ and is the last negativity to exhibit ESD, which occurs when $p \simeq .938$. For N_{14} the lowest eigenvalue is doubly degenerate and given by $\frac{1}{16}\left(-4 + 4p + p^2\right)$. ESD is exhibited at $p = -2 + 2\sqrt{2} \simeq .828$ which is the same dephasing value at which N_1 exhibits ESD. Again note the presence of bound entanglement for $.828 \leq p \leq .938$. The expectation value of the final state with respect to the entanglement witness, $\mathcal{W}_{C4H}$ is given by $\frac{1}{16}\left(-8\tilde{p} + p(8 + 4\tilde{p} - p)\right)$, Thus, the witness fails to detect entanglement for $p > 2(-\sqrt{2} + 2^{3/4}) \simeq .535$. The evolution of the above entanglement metrics as a function of p are shown in Fig. 1.

III. FINAL STATE FIDELITY

Having observed that some sort of ESD occurs for both of our chosen representations of the four qubit cluster state, we now seek to determine whether ESD affects the utilization of the cluster state as a means of implementing a general single qubit rotation in the measurement based cluster model of quantum computation. To implement such a rotation measurements at an angle θ_t with respect to the positive x axis in the $x-y$ plane are performed on the first three qubits, $t = 1, 2, 3$, giving a one qubit final state as a function of the measurement angles and

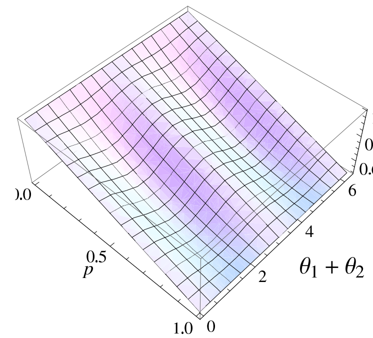

FIG. 2: (Color online) Fidelity of the state of the single unmeasured qubit from the four qubit cluster state $|C_4\rangle$ as a function of the dephasing strength p, and the sum of the first two measurement angles $\theta_1 + \theta_2$. The third measurement angle θ_3 does not affect the fidelity. The unmeasured qubit is the final state of the cluster computational logical qubit after performance of an arbitrary single qubit rotation via measurement. There is no sign of any sort of discontinuity that might have been expected due to ESD at $p \simeq .586$.

the dephasing strength, $\rho_f(p, \theta_1, \theta_2, \theta_3)$. We look at the fidelity of the state of the unmeasured qubit as compared to the same state without dephasing:

$$F_r(p, \theta_1, \theta_2, \theta_3) = \mathrm{Tr}[\rho_f(p, \theta_1, \theta_2, \theta_3)\rho_f(0, \theta_1, \theta_2, \theta_3)]. \tag{4}$$

For convenience we have assumed that the outcome of each measurement is -1 in the chosen measurement basis, such that $m = 0$ and no extra X rotations are necessary. A measurement of $+1$ would simply add the necessity for an X rotation. We note that the fidelity calculation was done only for initial states $|C_4\rangle$ and $|C_{4H}\rangle$ while full process tomography is needed to completely determine the dynamics of the single qubit rotation.

For initial state $|C_4\rangle$ the fidelity can be determined analytically,

$$\begin{aligned} F_{C4}(p, \theta_1, \theta_2, \theta_3) &= \frac{1}{4}(4 + p(p-3) \\ &+ p(1-p)\cos(2(\theta_1 + \theta_2))). \end{aligned} \tag{5}$$

Notice that for this representation, θ_3 cancels and the other measurement angles contribute only as $\theta_1 + \theta_2$. The fidelity is plotted in Fig. 2 and shows an oscillating plane steadily and smoothly decreasing toward, but never reaching, $F_{C4} = .5$. The amplitude of the oscillations decrease at high and low values of p and reach a maximum at $p \simeq .5$. We do not see any sort of sharp transition or discontinuity in the behavior of F_{C4} at $p \simeq .586$ as one might expect due to the sudden disappearance of N_{12} for the complete four qubit cluster.

As mentioned above, the initial state $|C_4\rangle$ undergoes ESD only with respect to N_{12}. One may suggest that the reason ESD is not manifest in the fidelity degradation of the unmeasured qubit for this initial state is because there is still some entanglement, N_1, which does not exhibit ESD, present in the state. To explore this we now

look at the initial state $|C_{4H}\rangle$ which, under dephasing, exhibits ESD for all negativity measures. Following the above, we find the fidelity of the final single qubit state as a function of p and measurement angles $\theta_t, t = 1, 2, 3$ for the intial state $|C_{4H}\rangle$ to be:

$$
\begin{aligned}
F_{C4H}(p,\theta_1,\theta_2,\theta_3) \;=\; & \frac{1}{64}(2\tilde{p}^3\cos(2(\theta_1-\theta_2)) + 4p'\cos(2\theta_2) + 2\tilde{p}^3\cos(2(\theta_1+\theta_2)) + 2p'\cos(2(\theta_1-\theta_3)) \\
+ \; & \tilde{p}^3\cos(2(\theta_1-\theta_2-\theta_3)) + 2p'\cos(2(\theta_2-\theta_3)) + \tilde{p}^3\cos(2(\theta_1+\theta_2-\theta_3)) \\
+ \; & 4(p-1)\cos(2\theta_1)\left(\tilde{p} - 2(p+1)\cos\theta_3^2\right) + 12p'\cos(2\theta_3) + 4(11 + 5\tilde{p}^3 + 3\cos(2\theta_3)) \\
+ \; & 2p'\cos(2(\theta_1+\theta_3)) + \tilde{p}^3\cos(2(\theta_1-\theta_2+\theta_3)) + 2p'\cos(2(\theta_2+\theta_3)) + \tilde{p}^3\cos(2(\theta_1+\theta_2+\theta_3)) \\
+ \; & 16\cos(2\theta_2)\cos\theta_3^2\sin\theta_1^2 + 8p^2(\cos\theta_3^2(1 + 2\cos(2\theta_2)\sin\theta_1^2) - \cos\theta_2\sin(2\theta_1)\sin(2\theta_3)) \\
+ \; & 8p(-4\cos\theta_3^2(1 + \cos(2\theta_2)\sin\theta_1^2) + \cos\theta_2\sin(2\theta_1)\sin(2\theta_3))).
\end{aligned}
\tag{6}
$$

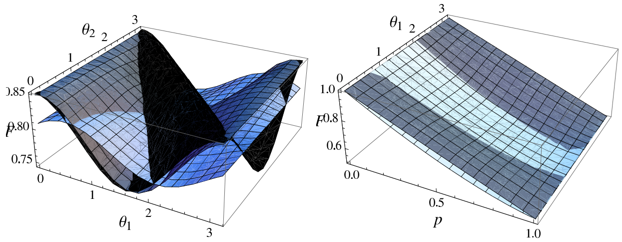

FIG. 3: (Color online) Left: using intial state $|C_{4H}\rangle$, fidelity of the state of the single unmeasured qubit such that an arbitrary rotation has been performed via the cluster state as a function of two of the measurement angles θ_1 and θ_2. The curves are $\theta_3 = \pi/16$ (gray) and $\pi/3$ (light) for $p = .3$. The black curve is the fidelity of the state of the single unmeasured qubit with dephasing for the intial state $|C_4\rangle$. This is plotted so as to compare the range of fidelities of the two initial states given the same evolution. Right: fidelity as a function of dephasing strength and θ_1 with $\theta_2 = \pi/4$ and the two curves again equal to $\theta_3 = \pi/16$ (gray) and $\pi/3$ (light). As a function of p we see the overall fidelity decreases steadily toward .5 without any discontinuity.

Fig. 3 plots the fidelity as a function of the three measurement angles and p (see figure caption). As a function of p the fidelity decreases almost uniformly approaching, but not reaching, $F_{C4H} = .5$. Again we do not see any discontinuity or change of behavior at the dephasing strengths where ESD is exhibited for the complete cluster state, $p \simeq .828$ and $p \simeq .938$.

Fig. 3 (left) also shows the fidelity of the state of the single unmeasured qubit for the intial state $|C_4\rangle$ and dephasing strength $p = .3$ as a function of the measurement angles. Note that the range of fidelity is the same for both initial states but the maximum and minimum points as a function of measurement angle are different. The equivalent fidelity range for the two cluster representations is in contrast to the disappearance of entanglement which occurs at different dephasing strengths for the two cluster state representations.

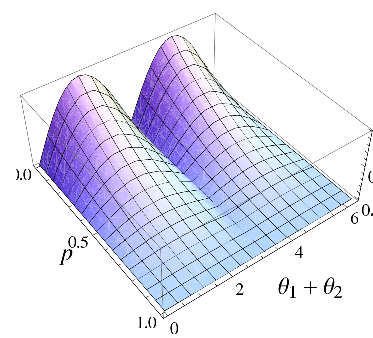

FIG. 4: (Color online) The concurrence between unmeasured qubits 3 and 4 after measurement on the first two qubits having started with the state $|C_4\rangle$. The concurrence is plotted as a function of dephasing strength and measurement axes (which contribute only as $\theta_1 + \theta_2$). There is no ESD exhibited for this concurrence function.

IV. TWO QUBIT FIDELITIES AND CONCURRENCE

So far our exploration of fidelity decay and entanglement as functions of dephasing indicate that ESD does not affect the utility of a cluster state as a means of implementing an arbitrary logical single qubit rotation. However, the picture changes when we explore fidelities and sudden bi-partite entanglement death of two qubits after having measured the other two qubits. To quantify the bi-partite entanglement between the two unmeasured qubits I use the concurrence [22], C_{jk}. The concurrence between two qubits j and k with density matrix ρ_{jk} is usually defined as the maximum of zero and Λ, where $\Lambda = \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}$ and the λ_i are the eigenvalues of $\rho_{jk}(\sigma_y^j \otimes \sigma_y^k)\rho_{jk}^*(\sigma_y^j \otimes \sigma_y^k)$ in decreasing order. σ_y^i is the y Pauli matrix of qubit i. For the purposes of clearly seeing at what point ESD occurs we will use Λ as the concurrence noting that ESD occurs when $\Lambda = 0$ in finite time (i. e. before $p \to 1$).

We start with the intial state $|C_4\rangle$, with measurements

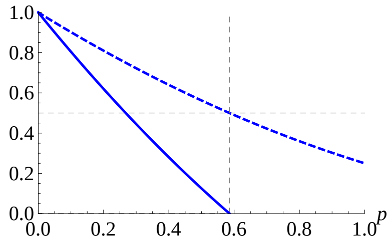

FIG. 5: (Color online) Fidelity (dashed line) of the state of qubits 2 and 4 after measurements on qubits 1 and 3 of the initial state $|C_4\rangle$ compared to the concurrence (solid line) between these same qubits. Note that the fidelity crosses .5 (horizontal light line) at $p \simeq .586$ (vertical light line) which is the same dephasing strength where ESD is exhibited by the concurrence between these two qubits and by N_{12} of $|C_4\rangle$.

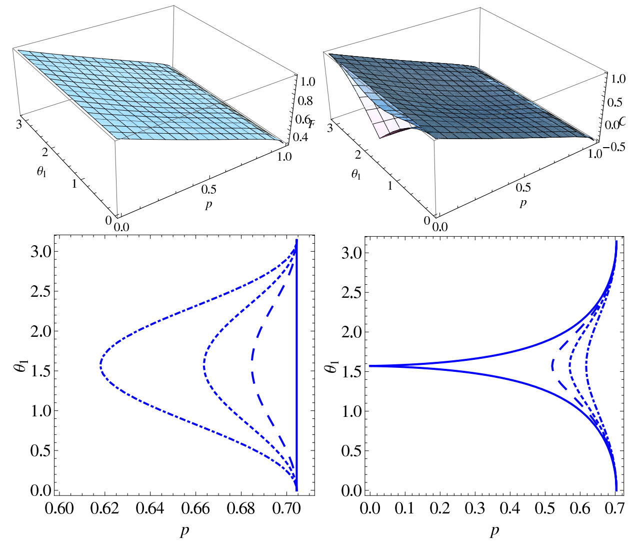

FIG. 6: (Color online) Fidelity (left) of the state of qubits 3 and 4 after measurement on qubits 1 and 2 and concurrence (right) between those qubits as a function of dephasing strength p and the measurement axes angles. Top left: fidelity as a function of p and θ_1 for $\theta_2 = \pi/4$. Bottom left: contours of fidelity equal to .5 for $\theta_2 = 0$ (chained line), $\theta_2 = \pi/4$ (dotted line), $\theta_2 = \pi/3$ (dashed line), and $\theta_2 = \pi/2$ (solid line). The fidelity in all cases converges to $p \simeq .704$ as θ_1 goes to zero. Top right: concurrence as a function of p and θ_1 for $\theta_2 = 0$ (bottom), $\theta_2 = \pi/4$ (middle), and $\theta_2 = \pi/2$ (top). Bottom right: contours of concurrence equal to zero showing where ESD occurs (values of θ_2 as in previous contour plot). The dephasing values at which ESD is exhibited approach .704, the exact value for which the fidelity goes to .5.

performed on qubits 1 (along the axis θ_1) and 2 (along the axis θ_2). The fidelity of the state of the two remaining qubits as a function of dephasing is given by Eq. (5), the fidelity of the final state of the fourth qubit after measurement on qubits 1, 2, and 3. This is so because eq. (5) does not depend on θ_3. The concurrence between unmeasured qubits 3 and 4 is a function only of the sum of the two measurement angles, $\theta_1 + \theta_2$, and p, and is given by $C_{34}^{C4} = \frac{1}{2\sqrt{2}}(\sqrt{A_{34} + B_{34}} - \sqrt{A_{34} - B_{34}})$ where:

$$\begin{aligned} A_{34} &= 2 + p(p-2)(p-1)^2 \\ &- (p-1)^2(2 + p(p-2)\cos(2(\theta_1 + \theta_2)), \end{aligned} \tag{7}$$

and

$$\begin{aligned} B_{34} &= 2(-2(p-1)^4(-1 + p(p-2) \\ &+ (p-1)^2\cos(2(\theta_1 + \theta_2)))\sin(\theta_1 + \theta_2)^2)^{1/2}. \end{aligned} \tag{8}$$

The concurrence is plotted in Fig. 4. We note that the fidelity of the state of the two unmeasured qubits never falls below .5 and no ESD is exhibited due to the dephasing.

If measurements are carried out on qubits 1 and 3 the fidelity of the state that remains on qubits 2 and 4 with dephasing is completely independent of any measurement angle and is given by $\frac{1}{4}(p-2)^2$. The concurrence between qubits 2 and 4 after the measurements is also independent of measurement angle and is given by $\frac{1}{2}(p^2 - 4p + 2)$. Note that the fidelity goes to .5 and the concurrence goes to zero at $p = 2 - \sqrt{2} \simeq .586$, the same value for which N_{12} of the four qubit cluster state exhibits ESD and the expectation value of the four qubit state with respect to $\mathcal{W}_{C4}$ goes to zero. While there is no discontinuity in the fidelity behavior at the dephasing strength that causes ESD, the fidelity does cross the critical value of .5 at the same dephasing strength. Thus, ESD indicates the severity of the decreased correlation between the dephased and not dephased state. The correlation between these metrics is shown in Fig. 5. Also note that in the previous case, where qubits 1 and 2 are measured, there is no exhibition of ESD and the fidelity never reaches .5.

Measurement on qubits 1 and 4 or qubits 2 and 3 give the exact same results as the measurements on 1 and 3.

We see similar correlations between fidelity and entanglement metrics when measuring cetain pairs of qubits of the inital state $|C_{4H}\rangle$. The fidelity of the state of qubits 3 and 4 upon measuring qubits 1 and 2 is given by:

$$\begin{aligned} F_{34}(p, \theta_1, \theta_2) &= \frac{1}{16}(8(1 + \tilde{p}) + p(-5 - 4\tilde{p} + p) \\ &- p(p-1)(\cos(2\theta_1) - 2\cos(2\theta_2)\sin\theta_1^2)). \end{aligned} \tag{9}$$

As shown in Fig. 6, when $p \simeq .704$ the fidelity goes to .5 as θ_1 approaches 0 or π or when θ_2 approaches $\frac{\pi}{2}$. This is also the maximum dephasing value for which we find ESD of the concurrence between unmeasured qubits 3 and 4 as shown in the figure (we do not have an analytical solution for the concurrence). Thus, while once again we do not have a change of fidelity behavior due to ESD, the sudden death of concurrence does indicate the lowering of fidelity to the critical value of .5.

The fidelity of the state of qubits 2 and 4 upon mea-

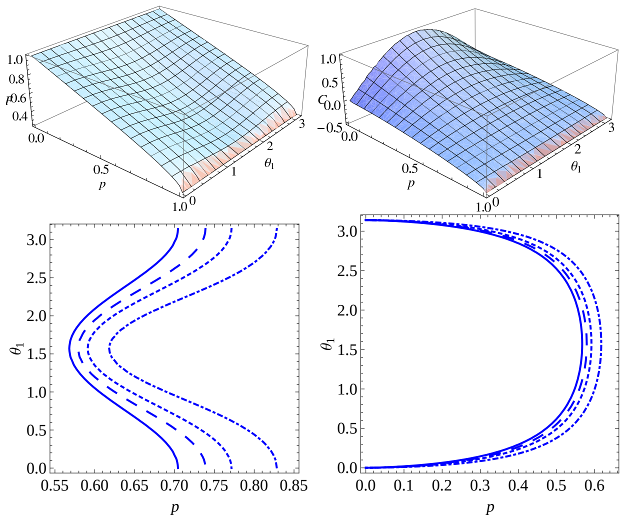

FIG. 7: (Color online) Fidelity (left) of the state of qubits 2 and 4 after measurement on qubits 1 and 3 and concurrence (right) between those qubits as a function of dephasing strength p and measurement axes angles. Top left: fidelity as a function of p and θ_1 for $\theta_3 = \pi/4$. Bottom left: contours of fidelity equal to .5 for $\theta_3 = 0$ (chained line), $\theta_3 = \pi/4$ (dotted line), $\theta_3 = \pi/3$ (dashed line), and $\theta_3 = \pi/2$ (solid line). As θ_1 and θ_3 go to zero the fidelity equals .5 contour goes to $p \simeq .828$, the value at which the state $|C_{4H}\rangle$ exhibits ESD for a number of entanglement measures. Top right: concurrence as a function of p and θ_1 for $\theta_3 = \pi/4$. Bottom right: contours of concurrence equal to zero showing where ESD occurs (values of θ_3 as in previous contour plot). The maximum dephasing value at which ESD is exhibited is .618.

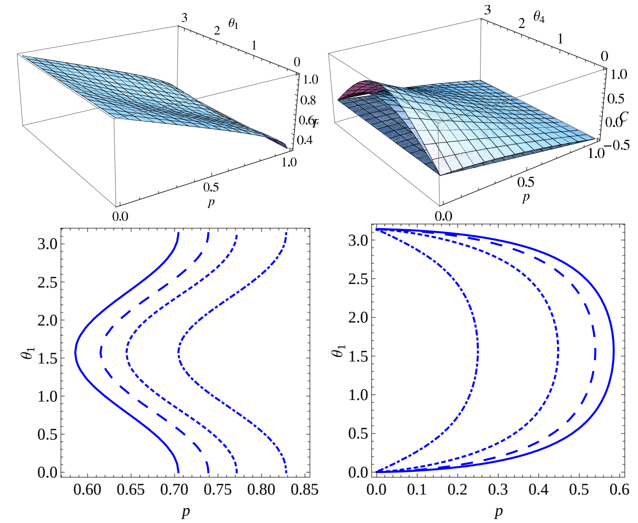

FIG. 8: (Color online) Fidelity (left) of the state of qubits 2 and 3 after measurement on qubits 1 and 4 and concurrence (right) between those qubits as a function of dephasing strength p and measurement axes angles. Top left: fidelity as a function of p and θ_1 for $\theta_4 = \pi/4$. Bottom left: contours of fidelity equal to .5 for $\theta_4 = 0$ (chained line), $\theta_4 = \pi/4$ (dotted line), $\theta_4 = \pi/3$ (dashed line), and $\theta_4 = \pi/2$ (solid line). As θ_1 and θ_4 go to zero the fidelity equals .5 contour goes to $p \simeq .828$, the value at which the state $|C_{4H}\rangle$ exhibits ESD for a number of entanglement measures. Top right: concurrence as a function of p and θ_4 for $\theta_1 = 0$ (bottom), $\theta_1 = \pi/4$ (middle), and $\theta_1 = \pi/2$ (top). Bottom right: contours of concurrence equal to zero showing where ESD occurs for $\theta_4 = \pi/32$ (chained line), $\theta_4 = \pi/8$ (dotted line), $\theta_4 = \pi/4$ (dashed line), and $\theta_4 = \pi/2$ (solid line). The maximum dephasing value at which ESD is exhibited is .586 which is the value at which ESD is exhibited for the state $|C_4\rangle$.

suring qubits 1 and 3 is given by:

$$\begin{aligned} F_{24}(p,\theta_1,\theta_3) &= \frac{1}{16}(8(1+\tilde{p}) + p(-5-4\tilde{p}+p) \\ &+ p\cos(2\theta_1)(1+2\tilde{p}-p) \\ &+ 2p\cos(2\theta_3)(\tilde{p}+(p-1)\sin\theta_1^2)) \end{aligned} \tag{10}$$

and is plotted in Fig. 7 along with the concurrence between unmeasured qubits 2 and 4. There does not appear to be a correlation between the fidelity and concurrence with respect to these two unmeasured qubits. However, the maximum p at which the fidelity crosses .5, when $\theta_1 = \theta_3 = 0$, is $2\sqrt{2}-2 \simeq .828$, the exact value where the four qubit state $|C_{4H}\rangle$ exhibits ESD for N_1, N_{13}, and N_{14}. The minimum value at which the fidelity crosses .5 is at .568. The maximum p at which ESD of concurrence is exhibited is .618.

The fidelity of the state of qubits 2 and 3 upon measuring qubits 1 and 4 is given by:

$$\begin{aligned} F_{23}(p,\theta_1,\theta_4) &= \frac{1}{16}(10+6\tilde{p}-p(7+2\tilde{p}-p)+\cos(2\theta_4) \\ &\times (-2+2\tilde{p}+3p-p^2)+2\cos(2\theta_1) \\ &\times (\tilde{p}-\tilde{p}^3\cos(2\theta_4)-(p-2)(p-1)\sin\theta_4^2)), \end{aligned} \tag{11}$$

and plotted in Fig. 8 along with the concurrence between unmeasured qubits 2 and 3. As in the previous case ESD may be an indicator of fidelity. The maximum p at which the fidelity crosses .5, which occurs for $\theta_1 = \theta_3 = 0$, is $2\sqrt{2}-2 \simeq .828$, the exact value where the four qubit state $|C_{4H}\rangle$ exhibits ESD for a number of entanglement measures. The minimum p at which the fidelity crosses .5 is $2-\sqrt{2} \simeq .586$, which is also equal to the the maximum p at which ESD of concurrence is exhibited. Though the initial state in this example was $|C_{4H}\rangle$ this is the value at which ESD occurs for the initial state $|C_4\rangle$. Such cross-correlation between the different cluster state representations can come from the measurements: measuring some of the qubits at certain angles transforms the state from one represenation to the other.

The fidelity of the state of qubits 1 and 4 upon measuring qubits 2 and 3 is given by:

$$\begin{aligned} F_{14}(p,\theta_2,\theta_3) &= \frac{1}{16}(10+6\tilde{p}-p(7+6\tilde{p}-p) \\ &+ (p-1)(-2+2\tilde{p}+p)(\cos(2\theta_2) \\ &+ 2\cos\theta_2^2\cos(2\theta_3)). \end{aligned} \tag{12}$$

The concurrence between unmeasured qubits 1 and 4 is

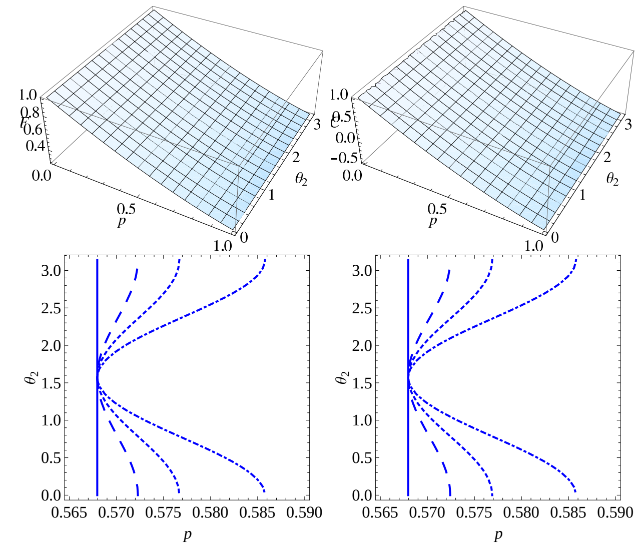

FIG. 9: (Color online) Fidelity (left) of the state of qubits 1 and 4 after measurement on qubits 2 and 3 and concurrence (right) between those qubits as a function of dephasing strength p and the measurement axes angles. Top left: fidelity as a function of p and θ_2 for $\theta_3 = \pi/4$. Bottom left: contours of fidelity equal to .5 for $\theta_3 = 0$ (chained line), $\theta_3 = \pi/4$ (dotted line), $\theta_3 = \pi/3$ (dashed line), and $\theta_3 = \pi/2$ (solid line). As θ_3 goes to $\pi/2$ the fidelity equals .5 contour goes to $p \simeq .568$, when θ_3 and θ_2 go to zero the fidelity equals .5 contour goes to $p \simeq .586$, the value at which the state $|C_{4H}\rangle$ exhibits ESD for a number of entanglement measures. Top right: concurrence as a function of p and θ_2 for $\theta_1 = \pi/4$. Bottom right: contours of concurrence equal to zero showing where ESD occurs (same θ_3 values as above). These curves are equivalent to those of the fidelity equals .5 curves.

given by $\mathrm{Max}[-\frac{\mathrm{p}}{2} - \mathrm{c}_{14}, -\frac{\mathrm{p}}{2} + \mathrm{c}_{14}]$ where

$$c_{14} = \frac{1}{4}((p-1)^2(16 + p(p-16) + p^2(\cos(2\theta_2) + 2\cos\theta_2^2\cos(2\theta_3)))^{1/2}. \qquad (13)$$

Fig. 9 demonstrates the strong correlation between the dephasing value where ESD is exhibited and the value where the fidelity goes to .5. Furthermore, the highest dephasing possible where ESD occurs (and when the fidelity goes to .5) is at $p = 2 - \sqrt{2} \simeq .586$, the same value for which we find ESD for the state $|C_4\rangle$. Again this points to the possibility of the measurement 'transforming' between the two representations of the cluster state. The lowest dephasing at which ESD occurs (or where the fidelity goes to .5) is at .568. All the above ESD and fidelity results are summarized in Table I.

Finally, we note that in all of the above we have made a number of assumptions. First, we have assumed that the intial cluster state is constructed perfectly. One way to relax this assumption is by looking at intial states of the form: $\rho_r = \frac{1-q}{16}\mathbb{1} + q|C_r\rangle\langle C_r|$, where $r = 4, 4H$. Preliminary explorations using this starting state indicate that there is merely a shift in entanglement values downwards but that there is no fundamental change in the behavior of the entanglement. Another assumption is that the dephasing strength is equal on all four qubits. This is unrealistic for a number of reasons but especially so if not all of the measurements are performed at the same time (non-simultaneous measurements are necessary when trying to implement a given logical rotation because the measurement axes for a given qubit depends on the outcome of the measurement on the previous qubit). A way to relax this assumption without significantly increasing the number of variables in the problem may be to add a $k\Delta p$ term to the dephasing strength where Δp represents the dephaing the occurs during the time between subsequent measurements and k is an integer.

V. CONCLUSIONS

In conclusion, I have studied the entanglement evolution of a four qubit (chain) cluster state in a dephasing environment. Specifically, I have looked at two represenations of the state differing by single qubit rotations. Both of these representations exhibit entanglement sudden death under sufficient dephasing. The difference in the dephasing strength at which this occurs may be important when deciding in what representation to store a cluster state. The issue of storage is especially relevant during the construction of optical cluster states but may have relevance to other implementations as well.

I asked whether ESD affects the utility of the cluster state in implementing a general single qubit rotation in the cluster state measurement based quantum computation paradigm. Judging from the fidelity decay of the single unmeasured qubit as a function of dephasing strength and the measurement axes angles of the three measurements the answer would seem to be no. I see no indication in the fidelilty behavior that ESD has taken place. Instead the fidelity decreases smoothly with increased dephasing with no discontinuities or dramatic changes in behavior. However, there are clear correlations (sometimes total and sometimes at certain limits) between the fidelity of the state of two qubits remaining from the four qubit cluster state after measurement on the other two qubits, and ESD of the negativity for the entire cluster state or ESD of the concurrence between the said two unmeasured qubits. This correlation does not appear as a discontinuity in the fidelity decay behavior but instead is manifest by the fidelity crossing the critical value of .5. Thus, we could say that ESD may be an *indicator* of how badly a certain cluster state operation was carried out. However, this is not the same as saying that ESD itself negatively affects quantum information protocols. The question of whether ESD affects quantum information protocols requires further study and may be related to the more general issue of the role of entanglement in quantum computation.

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TABLE I: The three parts of the table show 1) the values of p at which ESD occurs in four qubit cluster states subject to dephasing as measured by the expectation value of the proper entanglement witness ($\mathcal{W}_r$, $r = 4, 4H$), the negativity with partial transpose taken with respect to one (N_1) and two (N_{1k}, $k = 2, 3, 4$) qubits, 2) the value of p for which the concurrence, C_{jk}, between two unmeasured qubits j and k of the four qubit cluster state goes to zero after measurments on the other two qubits, 3) the value of p for which the fidelity of the dephased states goes to .5 for two qubit states F_{jk}, after measurement on the other two qubits, or the fidelity of the one qubit states F_r, after measurement on three qubits.

	$\mathcal{W}$	N_1	N_{12}	N_{13}	N_{14}
$\|C_4\rangle$	.586	none	.586	none	none
$\|C_{4H}\rangle$	.535	.828	.828	.938	.828
	C_{34}	C_{24}	C_{23}	C_{14}	
$\|C_4\rangle$	none	.586	.586	.586	
$\|C_{4H}\rangle$	$\leq .704$	$\leq .618$	$\leq .586$	$.568 \leq p \leq .586$	
	F_{34}	F_{24}	F_{23}	F_{14}	F_r
$\|C_4\rangle$	none	.586	.586	.586	none
$\|C_{4H}\rangle$	$.618 \leq p \leq .704$	$.568 \leq p \leq .828$	$.586 \leq p \leq .828$	$.568 \leq p \leq .586$	none

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